Example 2.3 Regression With Lagged Variables

In Example 1.25, we discovered that the Southern Oscillation Index (SOI) measured at time $t-6$ months is associated with the Recruitment series at time t, indicating that the SOI leads the Recruitment series by six months. Although there is evidence that the relationship is not linear (this is discussed further in Example 2.7), we may consider the following regression,

$$R_t = \beta_1 + \beta_2 S_{t-6} + w_t, \tag{2.26}$$

where R_t denotes Recruitment for month t and S_{t-6} denotes SOI six months prior. Assuming the w_t sequence is white, the fitted model is

$$\widehat{R}_t = 65.79 - 44.28_{(2.78)} S_{t-6} \tag{2.27}$$

with $\widehat{\sigma}_w = 22.5$ on 445 degrees of freedom. This result indicates the strong predictive ability of SOI for Recruitment six months in advance. Of course, it is still essential to check the the model assumptions, but again we defer this until later.

Performing lagged regression in R is a little difficult because the series must be aligned prior to running the regression. The easiest way to do this is to create a data frame that we call `fish` using `ts.intersect`, which aligns the lagged series.

```
fish = ts.intersect(rec, soiL6=lag(soi,-6), dframe=TRUE)
summary(lm(rec~soiL6, data=fish, na.action=NULL))
```

2.3 Exploratory Data Analysis

In general, it is necessary for time series data to be stationary, so averaging lagged products over time, as in the previous section, will be a sensible thing to do. With time series data, it is the dependence between the values of the series that is important to measure; we must, at least, be able to estimate autocorrelations with precision. It would be difficult to measure that dependence if the dependence structure is not regular or is changing at every time point. Hence, to achieve any meaningful statistical analysis of time series data, it will be crucial that, if nothing else, the mean and the autocovariance functions satisfy the conditions of stationarity (for at least some reasonable stretch of time) stated in Definition 1.7. Often, this is not the case, and we will mention some methods in this section for playing down the effects of nonstationarity so the stationary properties of the series may be studied.

A number of our examples came from clearly nonstationary series. The Johnson & Johnson series in Figure 1.1 has a mean that increases exponentially over time, and the increase in the magnitude of the fluctuations around this trend causes changes in the covariance function; the variance of the process, for example, clearly increases as one progresses over the length of the series. Also, the global temperature series shown in Figure 1.2 contains some evidence of a trend over time; human-induced global warming advocates seize on this as empirical evidence to advance their hypothesis that temperatures are increasing.

Perhaps the easiest form of nonstationarity to work with is the trend stationary model wherein the process has stationary behavior around a trend. We may write this type of model as

$$x_t = \mu_t + y_t \tag{2.28}$$

where x_t are the observations, μ_t denotes the trend, and y_t is a stationary process. Quite often, strong trend, μ_t, will obscure the behavior of the stationary process, y_t, as we shall see in numerous examples. Hence, there is some advantage to removing the trend as a first step in an exploratory analysis of such time series. The steps involved are to obtain a reasonable estimate of the trend component, say $\widehat{\mu}_t$, and then work with the residuals

$$\widehat{y}_t = x_t - \widehat{\mu}_t. \tag{2.29}$$

Consider the following example.

Example 2.4 Detrending Global Temperature

Here we suppose the model is of the form of (2.28),

$$x_t = \mu_t + y_t,$$

where, as we suggested in the analysis of the global temperature data presented in Example 2.1, a straight line might be a reasonable model for the trend, i.e.,

$$\mu_t = \beta_1 + \beta_2\, t.$$

In that example, we estimated the trend using ordinary least squares[3] and found

$$\widehat{\mu}_t = -11.2 + .006\, t.$$

Figure 2.1 shows the data with the estimated trend line superimposed. To obtain the detrended series we simply subtract $\widehat{\mu}_t$ from the observations, x_t, to obtain the detrended series

$$\widehat{y}_t = x_t + 11.2 - .006\, t.$$

The top graph of Figure 2.4 shows the detrended series. Figure 2.5 shows the ACF of the original data (top panel) as well as the ACF of the detrended data (middle panel).

[3] Because the error term, y_t, is not assumed to be iid, the reader may feel that weighted least squares is called for in this case. The problem is, we do not know the behavior of y_t and that is precisely what we are trying to assess at this stage. A notable result by Grenander and Rosenblatt (1957, Ch 7), however, is that under mild conditions on y_t, for polynomial regression or periodic regression, asymptotically, ordinary least squares is equivalent to weighted least squares.

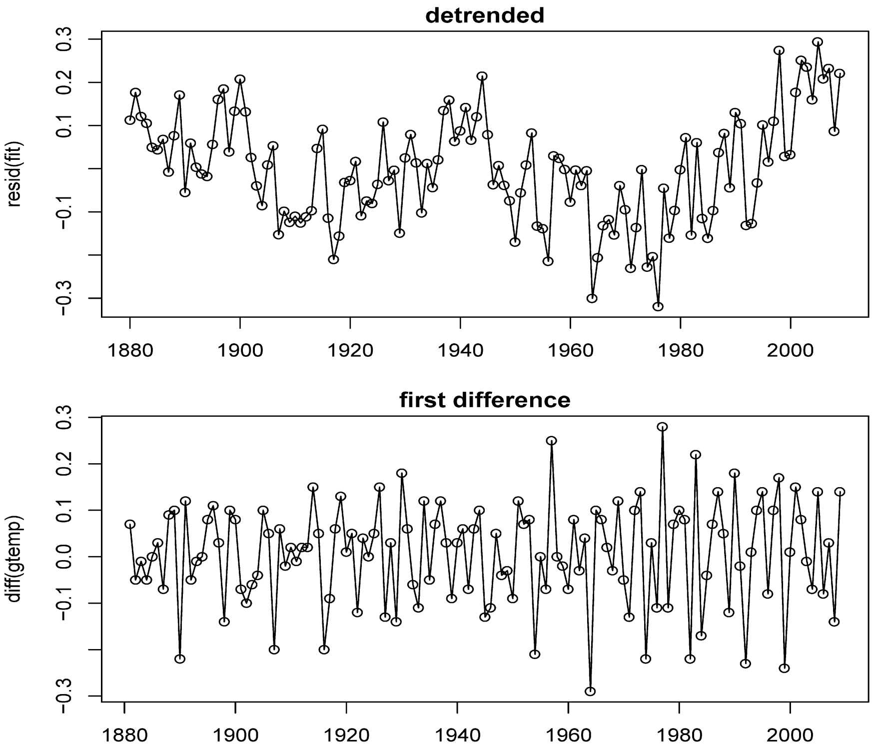

Fig. 2.4. Detrended (top) and differenced (bottom) global temperature series. The original data are shown in Figures 1.2 and 2.1.

To detrend in the series in R, use the following commands. We also show how to difference and plot the differenced data; we discuss differencing after this example. In addition, we show how to generate the sample ACFs displayed in Figure 2.5.

```
fit = lm(gtemp~time(gtemp), na.action=NULL) # regress gtemp on time
par(mfrow=c(2,1))
plot(resid(fit), type="o", main="detrended")
plot(diff(gtemp), type="o", main="first difference")
par(mfrow=c(3,1))  # plot ACFs
acf(gtemp, 48, main="gtemp")
acf(resid(fit), 48, main="detrended")
acf(diff(gtemp), 48, main="first difference")
```

In Example 1.11 and the corresponding Figure 1.10 we saw that a random walk might also be a good model for trend. That is, rather than modeling trend as fixed (as in Example 2.4), we might model trend as a stochastic component using the random walk with drift model,

$$\mu_t = \delta + \mu_{t-1} + w_t, \tag{2.30}$$

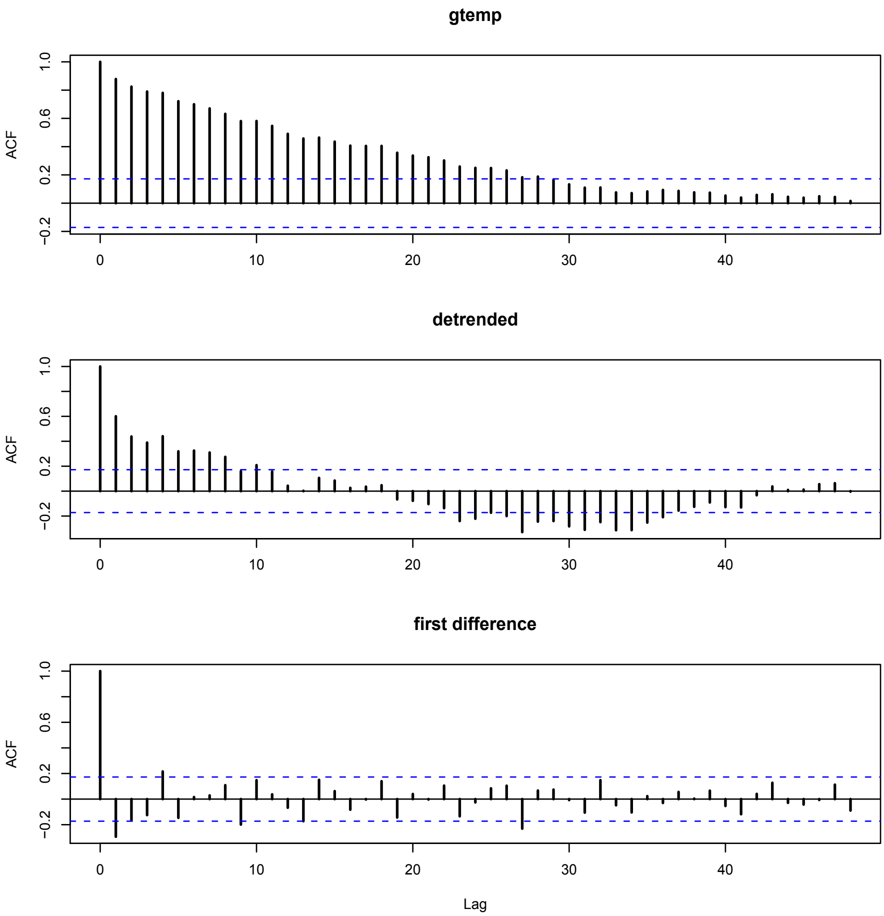

Fig. 2.5. Sample ACFs of the global temperature (top), and of the detrended (middle) and the differenced (bottom) series.

where w_t is white noise and is independent of y_t. If the appropriate model is (2.28), then differencing the data, x_t, yields a stationary process; that is,

$$\begin{aligned} x_t - x_{t-1} &= (\mu_t + y_t) - (\mu_{t-1} + y_{t-1}) \\ &= \delta + w_t + y_t - y_{t-1}. \end{aligned} \tag{2.31}$$

It is easy to show $z_t = y_t - y_{t-1}$ is stationary using footnote 3 of Chapter 1 on page 20. That is, because y_t is stationary,

$$\begin{aligned} \gamma_z(h) = \text{cov}(z_{t+h}, z_t) &= \text{cov}(y_{t+h} - y_{t+h-1}, y_t - y_{t-1}) \\ &= 2\gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1) \end{aligned}$$

is independent of time; we leave it as an exercise (Problem 2.7) to show that $x_t - x_{t-1}$ in (2.31) is stationary.

One advantage of differencing over detrending to remove trend is that no parameters are estimated in the differencing operation. One disadvantage, however, is that differencing does not yield an estimate of the stationary process y_t as can be seen in (2.31). If an estimate of y_t is essential, then detrending may be more appropriate. If the goal is to coerce the data to stationarity, then differencing may be more appropriate. Differencing is also a viable tool if the trend is fixed, as in Example 2.4. That is, e.g., if $\mu_t = \beta_1 + \beta_2\, t$ in the model (2.28), differencing the data produces stationarity (see Problem 2.6):

$$x_t - x_{t-1} = (\mu_t + y_t) - (\mu_{t-1} + y_{t-1}) = \beta_2 + y_t - y_{t-1}.$$

Because differencing plays a central role in time series analysis, it receives its own notation. The first difference is denoted as

$$\nabla x_t = x_t - x_{t-1}. \tag{2.32}$$

As we have seen, the first difference eliminates a linear trend. A second difference, that is, the difference of (2.32), can eliminate a quadratic trend, and so on. In order to define higher differences, we need a variation in notation that we will use often in our discussion of ARIMA models in Chapter 3.

Definition 2.4 *We define the* **backshift operator** *by*

$$Bx_t = x_{t-1}$$

and extend it to powers $B^2x_t = B(Bx_t) = Bx_{t-1} = x_{t-2}$, *and so on. Thus,*

$$B^k x_t = x_{t-k}. \tag{2.33}$$

It is clear that we may then rewrite (2.32) as

$$\nabla x_t = (1 - B)x_t, \tag{2.34}$$

and we may extend the notion further. For example, the second difference becomes

$$\begin{aligned} \nabla^2 x_t &= (1 - B)^2 x_t = (1 - 2B + B^2)x_t \\ &= x_t - 2x_{t-1} + x_{t-2} \end{aligned}$$

by the linearity of the operator. To check, just take the difference of the first difference $\nabla(\nabla x_t) = \nabla(x_t - x_{t-1}) = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2})$.

Definition 2.5 Differences of order d *are defined as*

$$\nabla^d = (1 - B)^d, \tag{2.35}$$

where we may expand the operator $(1-B)^d$ *algebraically to evaluate for higher integer values of* d. *When* $d = 1$, *we drop it from the notation.*

The first difference (2.32) is an example of a linear filter applied to eliminate a trend. Other filters, formed by averaging values near x_t, can produce adjusted series that eliminate other kinds of unwanted fluctuations, as in Chapter 3. The differencing technique is an important component of the ARIMA model of Box and Jenkins (1970) (see also Box et al., 1994), to be discussed in Chapter 3.

Example 2.5 Differencing Global Temperature

The first difference of the global temperature series, also shown in Figure 2.4, produces different results than removing trend by detrending via regression. For example, the differenced series does not contain the long middle cycle we observe in the detrended series. The ACF of this series is also shown in Figure 2.5. In this case it appears that the differenced process shows minimal autocorrelation, which may imply the global temperature series is nearly a random walk with drift. It is interesting to note that if the series is a random walk with drift, the mean of the differenced series, which is an estimate of the drift, is about .0066 (but with a large standard error):

```
mean(diff(gtemp))    # = 0.00659 (drift)
sd(diff(gtemp))/sqrt(length(diff(gtemp))) # = 0.00966 (SE)
```

An alternative to differencing is a less-severe operation that still assumes stationarity of the underlying time series. This alternative, called fractional differencing, extends the notion of the difference operator (2.35) to fractional powers $-.5 < d < .5$, which still define stationary processes. Granger and Joyeux (1980) and Hosking (1981) introduced long memory time series, which corresponds to the case when $0 < d < .5$. This model is often used for environmental time series arising in hydrology. We will discuss long memory processes in more detail in §5.2.

Often, obvious aberrations are present that can contribute nonstationary as well as nonlinear behavior in observed time series. In such cases, transformations may be useful to equalize the variability over the length of a single series. A particularly useful transformation is

$$y_t = \log x_t, \tag{2.36}$$

which tends to suppress larger fluctuations that occur over portions of the series where the underlying values are larger. Other possibilities are power transformations in the Box–Cox family of the form

$$y_t = \begin{cases} (x_t^\lambda - 1)/\lambda & \lambda \neq 0, \\ \log x_t & \lambda = 0. \end{cases} \tag{2.37}$$

Methods for choosing the power λ are available (see Johnson and Wichern, 1992, §4.7) but we do not pursue them here. Often, transformations are also used to improve the approximation to normality or to improve linearity in predicting the value of one series from another.